Optimization strategies for hardware-based cofactorization

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Jens Putzka Joachim von zur Gathen

b-it, partially funded by the BSI

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Goals

1. Optimize the COPACOBANA without changing the design.
2. After you have suceeded, analyze upcoming primitives like Edwards curves.

Contents

General optimizations

- Description of the model
- A toy example
- Further considerations

Optimizing the coordinate choices

- Edwards form
- Coordinate representations
- Use for the COPACOBANA

Conclusion

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Known factoring algorithms

- Best known factorization algorithm for large numbers (above 300 bits, say) is the General Number Field Sieve (GNFS).
- In the so called Cofactorization Step a large number of smaller integers have to be factored.
- For these factorizations LENSTRA's Elliptic Curve Method (ECM) is used.

Fact:

Many runs of the same algorithm can be efficiently realized on a dedicated hardware cluster.

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Components of a hardware cluster

Boards FPGAs ECM modules

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Small modules $\Longrightarrow$ more parallelism, less inputs

Components of a hardware cluster

Boards FPGAs ECM modules

Large modules $\Longrightarrow$ less parallelism, more inputs

Components of a hardware cluster

Boards FPGAs ECM modules

Mixed modules not allowed!

Optimize!

Design goals

We want:

- A generic model of the cluster.
- A fast algorithm that computes the optimal distribution.
- A way of measuring the runtime achievement...
- ... against what?!?

Specifically in the case of the GNFS we need:

- Estimates on the number of parallel units per FPGA.
- Estimates on the average cost of one run of the ECM.
- A mathematical model of the ouputs of the GNFS: Difficult!

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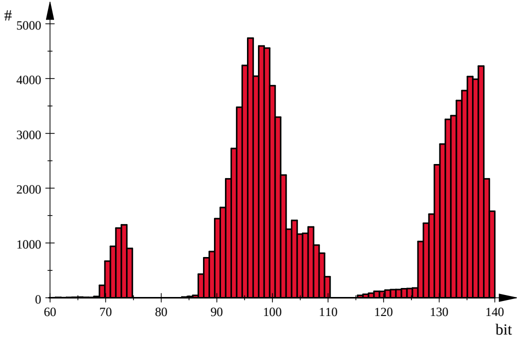

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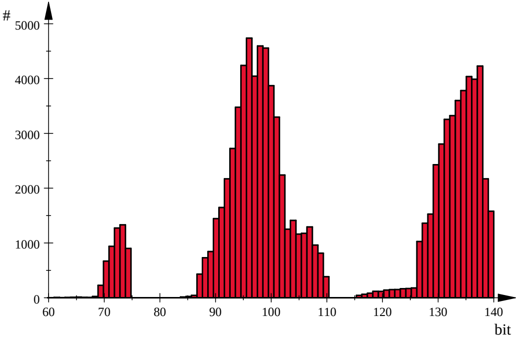

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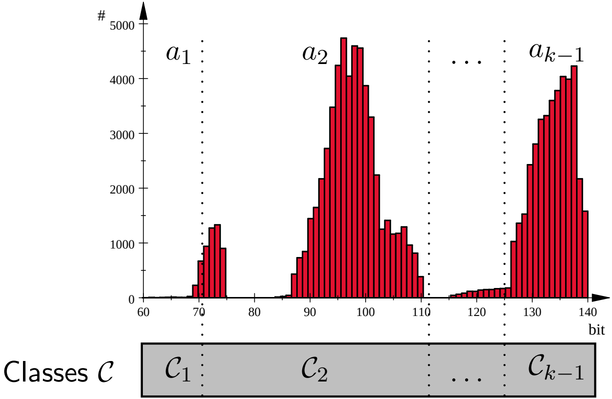

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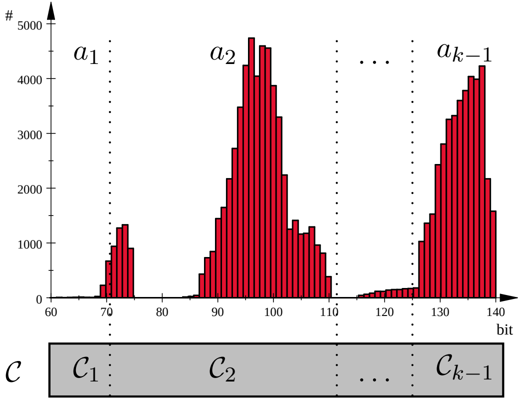

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\# ECM n_i

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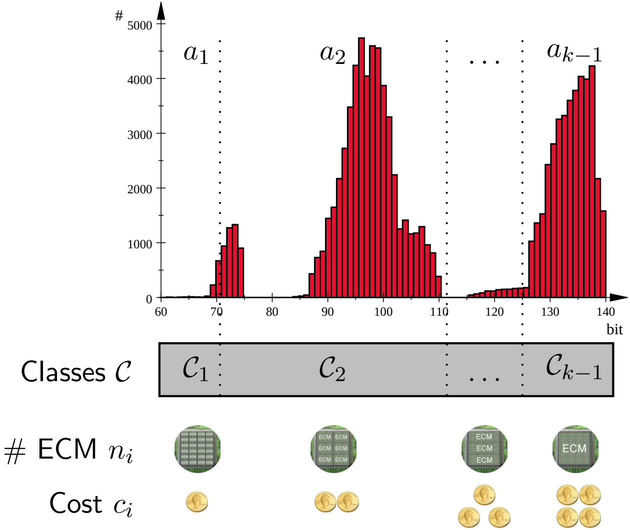

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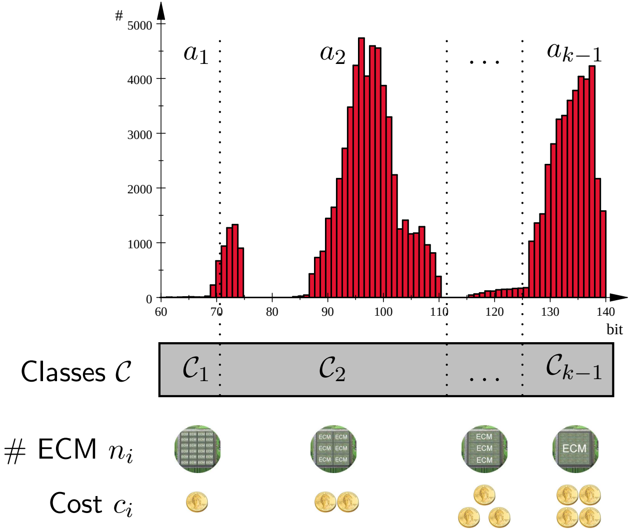

Optimize over all possible classes!

Distribution on FPGAs

Given classes we need to fill many FPGAs:

Classes $\mathcal{C}$ $\quad \mathcal{C}_1 \;\vdots\; \mathcal{C}_2 \;\vdots\; \ldots \;\vdots\; \mathcal{C}_{k-1}$

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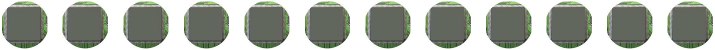

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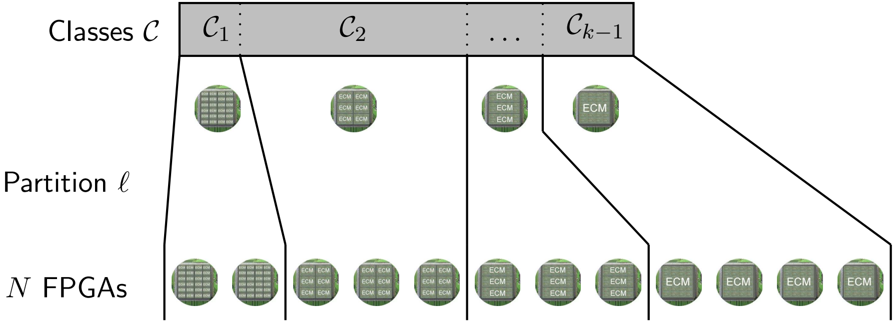

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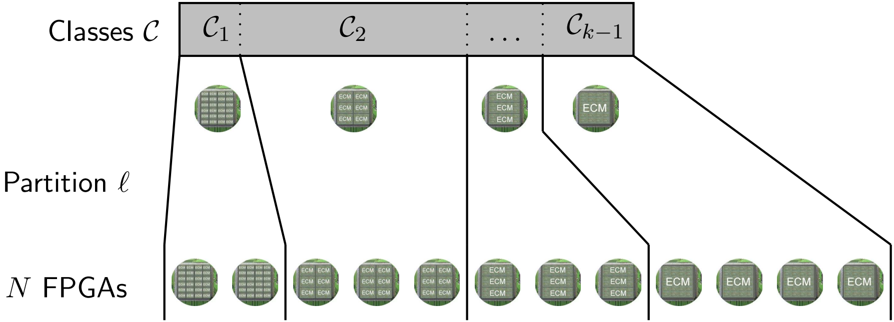

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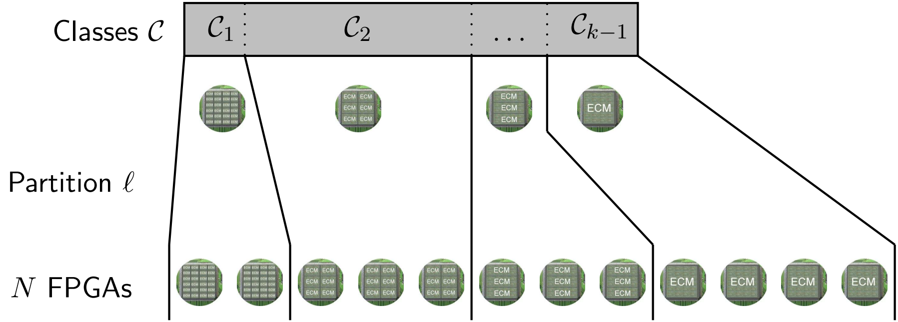

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Optimize over all configurations and classes, using e.g. BELLMAN's dynamic programming and a greedy heuristic!

A toy example

Assume we are constructing modules for $17i$-bit integers, given six data sets $\mathcal{D}_1, \ldots, \mathcal{D}_6$.

Number of parallel processes per chip:

Bitlength $17i$	51	68	85	102	119	136	153
Processes n_i	22	18	15	12	10	9	8

Average runtime of an ECM on the FPGA (in μs):

Bitlength $17i$	51	68	85	102	119
Cost c_i	856.35	1038.78	1221.22	1403.65	...

Optimize for 128 FPGAs.

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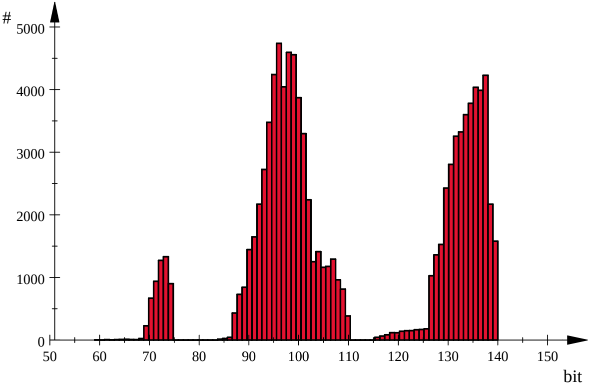
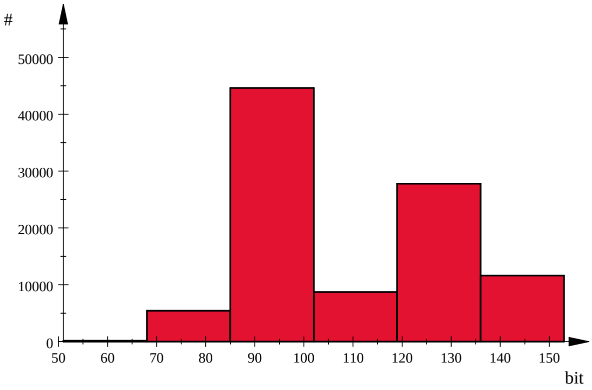
Input distribution for dataset $\mathcal{D}_1$:



Results of the optimization

Distribution on the classes:



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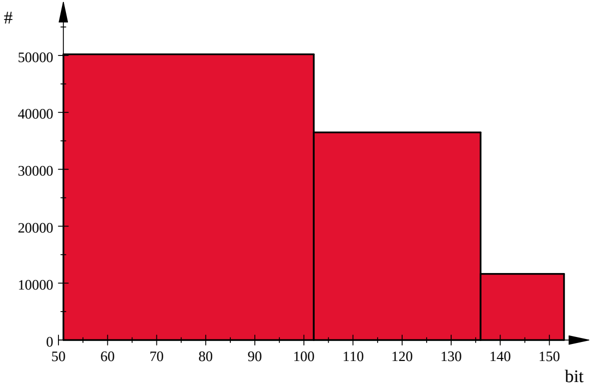

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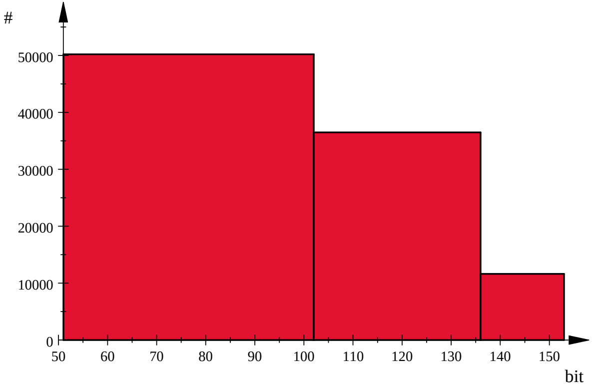

FPGAs: 47 58 23

Runtime gain

- Let $K \cdot g$ be the maximal size of the inputs.
- We compare the runtime to an unoptimized cluster.
- On such a cluster there are only modules for $K \cdot g$ bit numbers.
- The runtime is between

$$\sigma_{\mathcal{D}}^{-}(N,K) := \frac{1}{N}\sum_{i=1}^{K}\frac{c_i a_i}{n_K} \text{ and } \sigma_{\mathcal{D}}^{+}(N,K) := \frac{\#\mathcal{D}c_K}{N n_K}.$$

- Let $\gamma_{\mathcal{D}}^{\pm}$ denote the runtime gain of $\tau_{\mathcal{D}}$ against $\sigma_{\mathcal{D}}^{\pm}$ in percent.
- We obtain:

	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$
$\gamma_{\mathcal{D}}^{-}$	17.47	16.97	17.66	18.38	18.4	16.88
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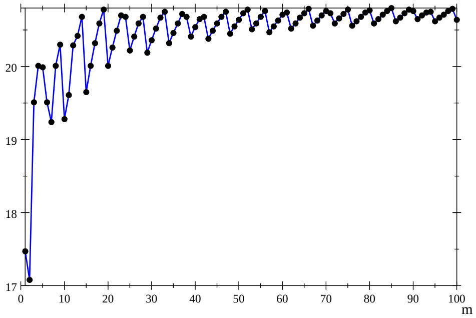
We can use our model to optimize m clusters in parallel, obtaining:



	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$
$\lim_{N\to\infty} \gamma_{\mathcal{D}}^{-}$	20.81	20.58	20.70	20.56	20.00	19.81
$\lim_{N\to\infty} \gamma_{\mathcal{D}}^{+}$	35.99	35.66	35.82	35.63	34.80	34.51

Online reconfiguration

- We have seen that the result of the optimization depends heavily on the input.
- On many clusters it is possible to reconfigure the FPGAs during the runtime of the cofactorization step.
- If we could use such a method, we could run a daemon on the controlling host computer that keeps track of the statistics of the inputs.
- The daemon has also to estimate when a reconfiguration makes sense.
- Further speedup possible!

To sum up

We have:

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Specifically in the case of the GNFS we have:

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Motivation

Edwards introduced in 2007 a new normal form for elliptic curves.

First Question

Do we have a speedup for addition and multiplication?

Second Question

Can these curves be used on the COPACOBANA?

Edwards curves

- Let K be a field of characteristic $p \neq 2$.
- Then many elliptic curves C are birationally equivalent to a curve of the form

$$x^2 + y^2 = 1 + dx^2y^2$$

where $d \notin \{0, 1\}$.
- If C is not equivalent to a curve in Edwards form, then its quadratic twist is.
- In this case one needs to quadratically enlarge K.

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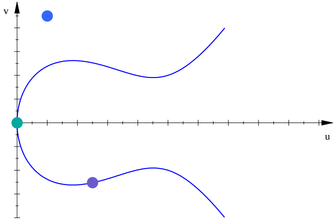
Transforming to Edwards form



$$\frac{v^2}{1-d} = u^3 + 2\frac{1+d}{1-d}u^2 + u$$

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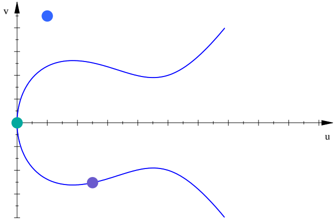

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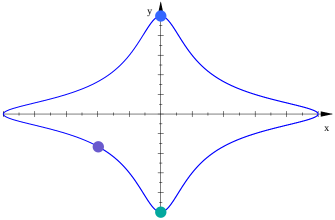

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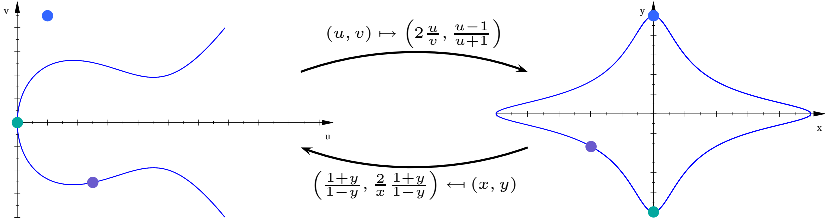

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$x^2 + y^2 = 1 + dx^2y^2$

Addition on Edwards curves

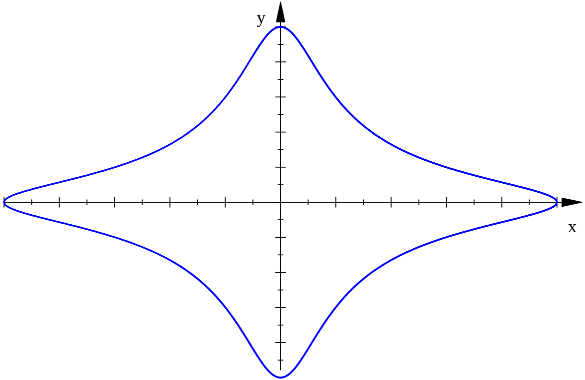
- We can define the following unified addition law:

$$(x_1, y_1) + (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right).$$

- Addition is well-defined if d is a nonsquare in the groundfield.
- The point $(0, 1)$ is the neutral element with respect to this addition law.
- Inverse of a point (x, y) is given by

$$-(x, y) = (-x, y).$$

Addition visualized



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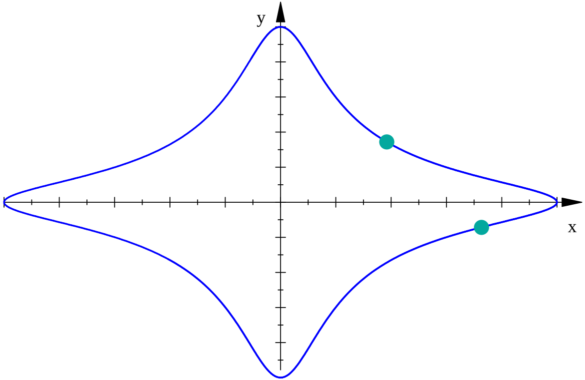

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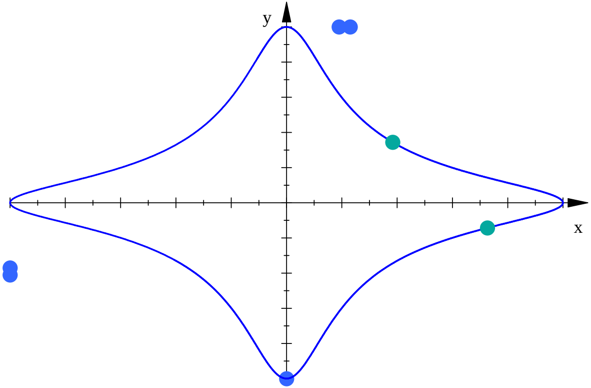

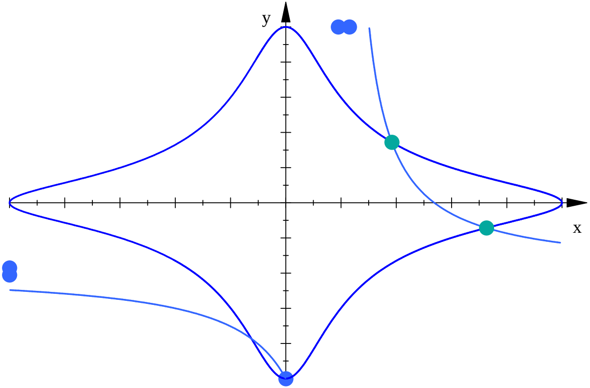
y
x

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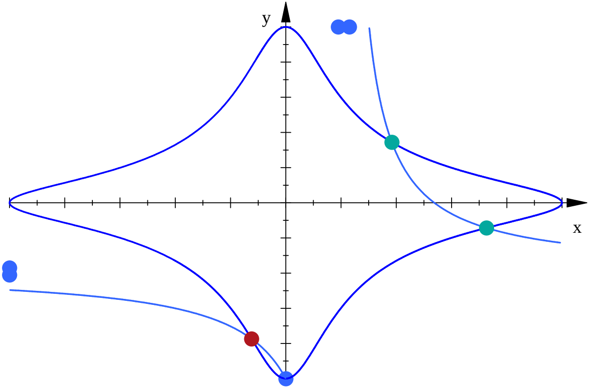

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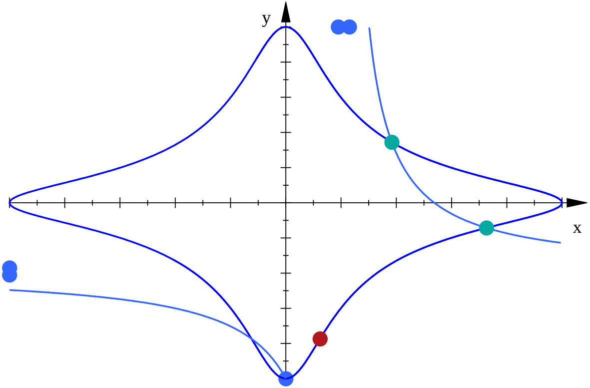

Twisted Edwards curves

- In 2008, Bernstein et al. introduced twisted Edwards curves.
- Prevents that we have to enlarge the ground field.
- A twisted Edwards curve $E_{a,d}$ of the curve E is given by the equation
$$ax^2 + y^2 = 1 + dx^2y^2.$$
- Here $a \neq d$ are both nonzero.
- Note that for parameters a_1, a_2, d_1, d_2 the curves E_{a_1,d_1} and E_{a_2,d_2} are quadratic twists of each other if $a_1d_2 = a_2d_1$.
- If further a_1/a_2 a square in K, then the curves are isomorphic, e.g. using the map
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Available coordinate choices

- Bernstein and Lange introduced different types of Edwards coordinates.
- Clearly this choice is crucial for the speed of the basic operations on the curve.
- We need to analyze the different types of coordinates to each other.
- Also we have to compare these to well-known coordinates (for classical elliptic curves).
- As usual: Count cost in terms of basic operations.

Operation in the ground field	Cost
Multiplication	$\mathbf{M}$
Squaring	$\mathbf{S}$
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Projective Edwards coordinates

- Affine point on an Edwards curve E is given by a tuple (x_1, y_1).
- Projectively the curve has the homogenized form

$$X^2Z^2 + Y^2Z^2 = Z^4 + dX^2Y^2$$

- Here the projective point $(X : Y : Z)$ with $Z \neq 0$ corresponds to the affine point $(X/Z, Y/Z)$.

Cost

General addition: $10\mathbf{M} + 1\mathbf{S} + 1\mathbf{D} + 7\mathbf{A}$.

Doubling: $3\mathbf{M} + 4\mathbf{S} + 6\mathbf{A}$.

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Speedup of one multiplication for general addition, but one more multiplication with a constant for doubling!

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Montgomery coordinates

- On a Mongomery curve $By^2 = x^3 + Ax^2 + x$ a point (x, y) is represented by a pair $(X\colon Z)$ such that $X/Z = x$.
- This representation does not distinguish (x, y) from $(x, -y)$!
- Thus $P + Q$ can only be computed if one knows

$$P, Q \text{ and } P - Q.$$

Cost

General addition: $6\mathbf{M} + 4\mathbf{S} + 1\mathbf{D}$.

Doubling: Same.

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Comparison to Edwards coordinates

- Montgomery curves have a faster differential point addition.
- Differential addition chains in general longer than standard ones.
- When using projective Edwards coordinates on twisted Edwards curves: Speedup possible.

Example

Scalar multiplication $[s]P$ with 256 bit scalar s:
Montgomery coordinates: On avg. $6\mathbf{M} + 4\mathbf{S} + 1\mathbf{D}$ per bit.
Edwards coordinates: On avg. $4.86\mathbf{M} + 4.12\mathbf{S} + 0.194\mathbf{D}$ per bit.
Speedup of $1.14\mathbf{M} - 0.12\mathbf{S} + 0.803\mathbf{D}$.

Comparison to Edwards coordinates

- Montgomery curves have a faster differential point addition.
- Differential addition chains in general longer than standard ones.
- When using projective Edwards coordinates on twisted Edwards curves: Speedup possible.

Example

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Use of Edwards coordinates for the COPACOBANA

- During the GNFS we will factor most of the time smallish numbers.
- For such numbers the Edwards form does not yet yield any speedup.
- Additionally: High development cost if one wants to redesign the ECM modules!
- Thus it seems that for the ECM on the COPACOBANA the use of Edwards coordinates does not (yet) make sense.

Problem

So what can we do?

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Problem

So what can we do?

Finding new constructions

Ideas:

- Exploiting the additional symmetry of Edwards curve, i.e. the operation $(x, y) \to (x, -y)$.
- Can we use existing constructions for Weierstraß curves in the case of Edwards curves?
- Can we improve the addition chains for ECM?

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Conclusion

Optimizations for the cluster:

- Independent of the implementation as long as it is scalable.
- Independent of the concrete coordinate choices.
- Already here a speedup of up to 30%.

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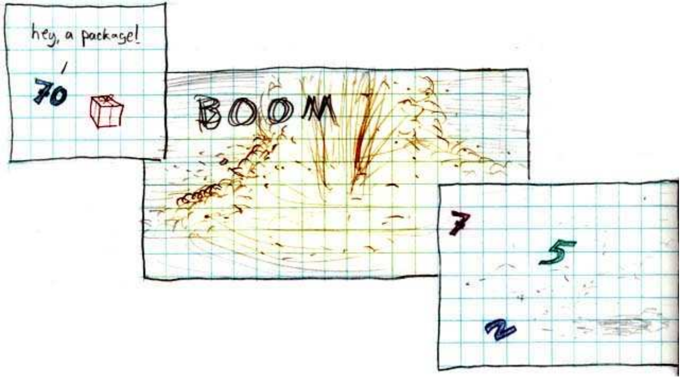

Thank you!