Striding Towards a New Subexponential Factoring Algorithm

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Outline

Motivation

- Understanding factorisation and especially why the Number Field Sieve is the best current factoring approach.

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- Understand why a more "natural " approach using the Riemann ζ function fails. Are we doomed to bang into a wall through an *analytic* approach?
- Get some money from RSA challenges (no current income).

Fermat's Idea

Suppose we can find x, y integers with $x^2 \equiv y^2 \pmod{N}$ and $x \not\equiv \pm y \pmod{N}$. Then $1 < \gcd(x - y, N) < N$ and this can be computed quickly, giving rise to a nontrivial factor of N.

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To find x and y, the most successful technique uses smooth numbers (divisible by "small" primes only). It is due to Morrison & Brillhart.
This idea is at the heart of the most successful factoring methods (QS and NFS), except ECM.

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- QS: $\exp\left((c_1 + o(1))(\log N)^{1/2}(\log\log N)^{1/2}\right)$
- ECM: $\exp\left((c_2 + o(1))(\log p)^{1/2}(\log\log p)^{1/2}\right)$, (where p is smallest prime dividing N)
- NFS: $\exp\left((c_3 + o(1))(\log N)^{1/3}(\log\log N)^{2/3}\right)$

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We present an approach which is likely to yield

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- Translates an arithmetic problem into an analytic one.
- All running times are proven, no assumptions!
- Much room for future improvements.

Approaching Multiplicative Functions

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How do we find a good approximation to $\phi(N)$?

First Attempt with Riemann

Riemann zeta function is

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \quad \Re s > 1$$

It can be continued to a meromorphic function in $\mathbb{C}$ with simple pole with residue 1 at $s = 1$. Also

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n \geq 1} \frac{\phi(n)}{n^s} \quad \Re s > 2$$

Isolating $\phi(N)$

Classical: Compute $\Phi(x) = \sum_{n<x} \phi(n)$ by

$$\Phi(x) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} \frac{\zeta(s-1)}{\zeta(s)} \frac{x^s}{s} \, ds$$

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and move line of integration "to the left". Problem: we hit the Riemann zeros, spooky beings! Can we avoid them?

Second Attempt with Riemann

We now consider $\sigma(N) = N + 1 + p + q$. As before, a close approximation to $\sigma(N)$ will reveal p. Here

$$\zeta(s)\zeta(s-1) = \sum_{n \geq 1} \frac{\sigma(n)}{n^s} \quad \Re s > 2$$

and hence if $S(x) = \sum_{n<x} \sigma(n)$ we get

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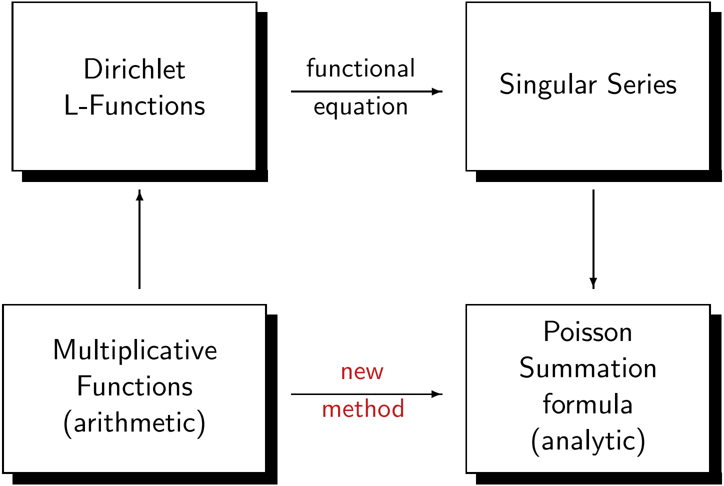
Problem: $|\zeta(s)| \approx |s|^{(1-\Re s)/2}$ as $|\Im s| \to \infty$ so cannot move the line of integration far enough to the left (to $\Re s \leq 0$)

The Mellin Transform Approach



Dirichlet L-functions

Let ℓ be prime and $\chi\colon (\mathbb{Z}/\ell)^* \to \mathbb{C}^*$ be a homomorphism, called Dirichlet character modulo ℓ. We define $\chi(n) = 0$ if ℓ divides $n \in \mathbb{Z}$. The Dirichlet L-function associated to χ is

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \ , \quad \Re s > 1$$

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It is an entire function if χ is not the trivial character, as we will henceforth suppose.

Dirichlet L-functions (cont'd)

We let $r \geq 2$ and $\beta_m \in \mathbb{C}$ "fixed". Define

$$\sigma_r(n) = \sum_{d_1 d_2 \cdots d_{r-1} | n} d_1^{\beta_1} d_2^{\beta_2} \cdots d_{r-1}^{\beta_{r-1}}$$

Then

$$L(s,\chi)L(s-\beta_1,\chi)\cdots L(s-\beta_{r-1},\chi) = \sum_{n=1}^{\infty} \frac{\sigma_r(n)\chi(n)}{n^s}$$

Mellin Transform

We let $\nu \in \mathbb{C}$ with $\Re\nu > 1$. Define

$$f_\nu(t) = \begin{cases} (1-t)^{\nu-1} & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases}$$

The Mellin transform of f_ν is

$$\frac{\Gamma(\nu)\Gamma(s)}{\Gamma(\nu+s)} = \int_0^\infty f_\nu(t) t^{s-1}\, dt$$

Inverse Mellin Transform

We have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} L(s,\chi)L(s-\beta_1,\chi)\cdots L(s-\beta_{r-1},\chi)\frac{\Gamma(\nu)\Gamma(s)}{\Gamma(\nu+s)}\,x^s\,ds$$
$$=\sum_{n\leq x}\sigma_r(n)\chi(n)f_\nu\left(\frac{n}{x}\right)$$

Call the right-hand side

$$F(\nu)=\sum_{n\leq x}\sigma_r(n)\chi(n)\left(1-\frac{n}{x}\right)^{\nu-1}$$

Isolating p dividing N

We estimate

$$F^{(k)}(\nu) = \sum_{n \le x} \sigma_r(n)\chi(n)\left(1 - \frac{n}{x}\right)^{\nu-1} \log^k\left(1 - \frac{n}{x}\right)$$

If $x = N + \dfrac{1}{N^2}$ we get

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If $x = N + \dfrac{1}{N^2}$ we get

$$F^{(k)}(\nu) = (-3\log N)^k \sigma_r(N)\chi(N)N^{-3(\nu-1)} + O(N^3(2\log N)^{k+r})$$

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Choosing $k > c_1 r \log N$ and supposing we can compute $F^{(k)}(\nu)$ with good precision we get a value for $\sigma_r(N)$ up to an error $O(N^{-c_2})$, where $c_2 \to \infty$ as $c_1 \to \infty$. If $N = pq$, then as before this is sufficient to obtain p.

The Functional Equation

The Dirichlet L-function of a primitive character χ of modulus $\ell > 1$ is an entire function satisfying the functional equation (given here in asymmetric form)

$$L(s,\chi) = \frac{1}{2\pi i}\left(\frac{2\pi}{\ell}\right)^s \tau(\chi)\Gamma(1-s)L(1-s,\bar{\chi}) \cdot \left(e^{i\pi s/2} - \chi(-1)e^{-i\pi s/2}\right)$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{\ell} \chi(m)\exp(2\pi i m/\ell)$$

New Identities

Moving the line of integration to the left and using the functional equation shows

$$F(\nu) \approx R + \frac{\tau(\chi)^r \left(\frac{2\pi i}{\ell}\right)^{r-\beta_1-\cdots-\beta_{r-1}} \Gamma(\nu)}{(2\pi i)^{r+1}} x(\cos \pi\nu - \sin \pi\nu)$$

$$\times \int_{(1+1/r)} \left\{ \left(\frac{2\pi i}{\ell}\right)^r x \right\}^{-s} \Gamma(s-\nu)\Gamma(s+\beta_1)\cdots\Gamma(s+\beta_{r-1})$$

$$L(s,\bar\chi)L(s+\beta_1,\bar\chi)\cdots L(s+\beta_{r-1},\bar\chi)\, ds$$

where R is some residue, independent of x and ν. In view of the previous expression, it is appropriate to choose $\ell \approx x^{1/r}$ so that the integral does not depend on x (hence N).

The Singular Series

Using the multiplication theorem

$$\Gamma(s)\Gamma\left(s+\frac{1}{r}\right)\Gamma\left(s+\frac{2}{r}\right)\cdots\Gamma\left(s+\frac{r-1}{r}\right) = (2\pi)^{(r-1)/2} r^{1/2-rs}\Gamma(rs)$$

we arrive at evaluating terms for $F^{(k)}(\nu)$ which consist of derivatives of $\Gamma(\nu)(\cos\pi\nu - \sin\pi\nu)$ times the following series

$$\frac{1}{\tau(\chi)}\sum_{m=1}^{\ell}\chi(m)\sum_{d_1,\ldots,d_r\geq 1}\frac{d_1^{-\beta_1}d_2^{-\beta_2}\cdots d_{r-1}^{-\beta_{r-1}}\log^j(d_1\cdots d_r)}{(d_1\cdots d_r)^{\frac{1}{2r}-\frac{\beta_1+\cdots+\beta_{r-1}-\nu}{r}}} \cdot e^{2\pi i\left(\frac{m}{\ell}d_1\cdots d_r+(d_1\cdots d_r)^{1/r}\right)}$$

Computing the Singular Series

The singular series can be written as

$$S_j = \sum_{d_1,\ldots,d_r \geq 1} \frac{\log^j d_r}{d_1^{a_1} d_2^{a_2} \cdots d_r^{a_r}} e^{2\pi i\left(\beta d_1 \cdots d_r + \gamma (d_1 \cdots d_r)^{1/r}\right)}$$

Approximate the singular series as

$$\sum_{(d_1,\ldots,d_r)\in\mathbb{Z}^r} f(d_1,\ldots,d_r) = \sum_{(\delta_1,\ldots,\delta_r)\in\mathbb{Z}^r} \hat{f}(\delta_1,\ldots,\delta_r)$$

by Poisson summation. Here f is a $C_c^\infty(\mathbb{R}^r)$ function interpolating the summands of the singular series so that its Fourier transform $\hat{f}$ is decreasing super-polynomially. We therefore need to compute only $O(N^\epsilon)$ terms with precision $O(N^{-c})$, which should be possible in polynomial time.

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- Write up this step

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- Ultimately, this shows that factoring could be done in $O(N^{1/r})$ for any r (subexponential). But in this last step with need to sample at least 2^r points. Therefore, we can only negotiate $r \approx \sqrt{\log N}$ and runtime is $O\left(e^{\sqrt{\log N}}\right)$ at best (naive estimate).

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- Need to implement it in practice.

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A New Factoring Algorithm

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- Introduction
- Heuristics
- Factoring with L-functions
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- Advantage is that it transforms the arithmetic problem of factoring N into a purely analytic one (evaluation of the singular series, "independent" of N).
- Should lead to a deterministic factoring algorithm with proven running time $O(\exp(c_1\sqrt{\log N \log\log N}))$
- Hoping to extend this to $O(\exp(c_3(\log N)^{1/3}(\log\log N)^{2/3}))$

Thank You!